Philosophy of science

An introduction

John-Jules Ch. Meyer

ICS - UU

A classification of sciences

- Formal sciences
  - study of 'pure' structures
    - *mathematics and logic*
- Empirical sciences
  - study of "maps of the concrete world"
    - *natural sciences*
    - *cognitive sciences*
    - *cultural and social sciences (including linguistic studies)*

A classification of sciences

- Practical sciences
  - study of the applications themselves
    - *technical sciences*
    - *study of law*
    - *health sciences*
    - *economics*
    - *management studies*
    - *pedagogical studies*
    - *theatre studies*

Formal sciences

- study *'consistent' theories of abstract structures*, not tied to one single area of science, but applicable in diverse more concrete contexts
- the ontological status of the structures studied themselves is subject of much philosophical discussion ((possibly) real versus imaginary)

Formal sciences

- these theories are *formal* in the sense that they concern *form* rather than *content* (concrete interpretation)
- typical examples are *computational* en *reasoning systems* like in algebra and logic: these work with *formal languages* that are gemanipulated on a purely *formal (syntactic)* basis (*'formal game'*)
- E.g. the tautology p ¬p holds *independent* from the *(concrete) interpretation* of p

Formal sciences

- *non-empirical*: structures are constructed, but no maps of the world / reality
- advantages of formal approaches:
  - very *precise / well-defined*
  - 'computation on *representations* instead of *interpretations*'

Empirical sciences

- yield *'maps of the concrete world'*
- In itself *non-formal*, although formal sciences / means (from mathematics and logic) may be employed to analyze results and theories obtained from *empirical material* further

Empirical sciences

- the general method connects *facts* and *hypothesis / theory*
- *the empirical cycle*: observation □ theory □ observation □ theory □ observation □ ...

Practical sciences

- Not only being used in practice, but also *the application itself is subject of scientific research*

Computer science as a science

- comp.sc. as a *formal science*
  - Use of *formal languages and methods* for the *specification (and verification)* of software / systems, such as e.g. Hoare logic and process algebra
  - *(complexity) analysis* of algoritms

Computer science as a science

- comp. sc. as a *experimental / empirical science*
  - *testing and debugging* software / systems
  - *testing performance* of softw / systems
  - the use of *experimental architectures*, such as *neural networks, agent systems*

Computer science as a science

- comp. sc. as a *technical / practical science*
  - the *design and construction* of software / systems

Philosophy of mathematics

Philosophical positions in mathematics

- logicism (Frege, Russell)
  - "mathematics is a branch of *logic*"
- formalism (Hilbert)
  - "mathematics is the science of *formal systems*"
- intuitionism (Brouwer)
  - "mathematics is about *mental constructions*"

Philosophical positions in mathematics

- Making these philosophical positions precise has contributed to the birth of the area of *mathematical logic*.
- The immediate cause for researching the foundations of mathematics was the discovery of *paradoxes* in certain fundamental parts of mathematics, in particular *set theory*

(Naïve) Set Theory

- attempt of defining sets from ∅ by means of operations such as
  - union ∪,
  - power set P(.) and
  - the *full comprehension principle*.

Full comprehension principle

- *full comprehension principle*: for every well-formulated condition P(x) there exist a set V that exactly contains the elements x satisfying P(x)
  - $V = \{ x \mid P(x) \}$
- this definition appeared to be too *vague* and gave rise to serious *paradoxes*

Cantor's paradox

- Let S be the set of all sets, and T the set of all subsets of S.
- Then Cantor's theorem says that cardinality(S) < cardinality(T).
- On the other hand T is a subset of S, the set of *all* sets. Thus cardinality(T) ≤ cardinality(S). Contradiction.

Russell's paradox:

- Let R = {V I V □ V}. According to Cantor this is a well-defined set. However: we have that R □ R □ R □ R.

  - Russell's pseudo-paradox of the barber: consider a barber shaving all people who do not shave themselves. The barber shaves himself iff he does not shave himself.

Solution to the paradoxes of set theory

- Elimination of 'too big sets' by means of *axioms*.
- naive set theory □ *formal-axiomatic set theory*
- e.g. the system ZF (Zermelo-Fraenkel), probably the simplest system in which most of the existing mathematics can be derived but not the paradoxes, as far as is known thusfar...

The system ZF

- *System ZF* contains axioms such as:

□ □x (Vx □ □y ¬(y □ x))

(existence of empty set ∅)

□ □x□y((Vx □ Vy □ □z (z □ x □ z □ y)) □ x = y)

(extensionality)

□ □x(Vx □ □y(Vy □ □z(z□y□ □w(w□x □ z□w))))

(union: y = □ x)

A few more axioms of ZF

□ □x(Vx □ □y(Vy □ □z (z □ y □ (Vz □ z □ x))))

(powerset: y = Px)

□ □x (Vx □ ∅□ x □ □y(y □ x □ {y} □ x))

(existence infinite set)

Logicism

- instance of so-called *Platonic Realism*
- mathematical objects 'exist' independent from the mathematician
- all mathematical notions reducible to abstract properties
- mathematics is the study of the *logical* (evident) basic principles wrt these properties
- *mathematics is a branch of logic*

Logicism

- *Russell* tried to perform this reduction to logic in *Principia Mathematica*.
- This attempt was *not entirely successful:*
- In order to avoid the paradoxes a further *complication* was needed (*'theory of types' + 'axiom of reducibility'*)
- All this weakened the claim of the logicists that mathematics can be reduced to logic substantially; it comes down to *mathematics = logic + set theory*

Formalism

- mathematics is *manipulating finite configurations symbols*, according to certain *rules*
- mathematics is the science of *formal systems,* consisting of a well-described *syntax* and a *derivation criterium*
- *N.B. mathematics itself is NO formal system; it only studies formal systems*

Formalism

- *Configurations*:
  - some have concrete meaning;
  - some other are *meaningless*
- Choice of *rules*: out of *pragmatic* reasons □ concrete sensible / useful derivations
  - vb. *Predicate logic, formal arithmetic, ...*

Formalism

- acceptance of the fact that parts of classical mathematics which use the *'completed' infinite* go beyond what is intuitively evident □ focus on *'certain' core of mathematics which can be axiomatised formally*
- *Core problem*: how can one prove parts of mathematics *consistent*, in another way than using models comprising apparently *unreliable sets* (*relative* consistency)?

Hilbert's programme: the metamathematical method

- Central problem: *absolute consistency proofs*
- Is (mathematics□) arithmetic consistent?
- To answer this question Hilbert proposed to employ a *certain evident kind of reasoning* (so-called *finitistic* methods): of *elementary combinatorial* nature, such as simple arithmetical operations and checking decidable properties.

Finitistic methods

- *finitistic mathematics* was regarded by Hilbert as the *true mathematics*: allow concrete representation + manipulation of sequences of tokens / symbols; is part of arithmetic (after coding)
- *properties of formalized mathematics* should be proven in the *meta language* via *finitistic* methods ('*metamathematica*')

Consistency proofs

- a *finitistic* proof of the *consistency* of arithmetic ($\vdash_{fin}$ Con_{PA} , where Con_{PA} stands for the expression ¬□x Prov(x, □0=1□) with x a code of a proof and □0=1□a code for the assertion 0=1) would then guarantee the consistency of arithmetic : $\vdash_{PA}$ Con_{PA}

Intuitionism (Brouwer)

- mathematics is a *stand-alone activity* concerning *mental constructions according to selfevident rules, independent from language.*
- Gave rise to critical review of

  □ the notion of an *(existence) proof*

  □ the notion of a *computable function*

J.-J. Ch. Meyer

Intuitionism: infinite sets

- According to Brouwer the *positive integers* constitute the starting point of mathematics via repeated duplication of the element 'I': 'I', 'II', 'III', ... -- having to do with the notion of *'time'*
- *infinite sets* are intuitionistically always *potentially* infinite (*under construction*) rather than *actually* infinite

J.-J. Ch. Meyer

Truth in intuitionism

- Truth of a proposition must be *constructive* : it must rest on *proof* (a certain kind of *mental construction*) ---> consequences for proofs of □-propositions
- <u>Consequence</u>: *propositions aren't true or false;* they can also be *undetermined*; even *inherently* so, if it concerns a *undecidable* property:

  $\not\models_{intuit}$ p ¬p

J.-J. Ch. Meyer

Intuitionistic logic

- the intuitionistic interpretation (of truth) of propositions gives rise to a *non-classical logic:*
- *So-called intuitionistic logic (Heyting)*
- *Propositions are*: 'reports of completed *proofs'*

J.-J. Ch. Meyer

Intuitionistic logic

- truth conditions:

  □ P Q : at least one of P, Q is *proven*

  □ P □ Q : both P and Q is *proven*

  □ P □ Q : a construction C is available of which it is *proven* that if C is applied to any possible *proof* of P, the result is a *proof* of Q

J.-J. Ch. Meyer

Intuitionistic logic

- ¬P : is the same as "P □ □", i.e. any possible *proof* of P can be transformed into a *proof* of a contradiction
- □x P(x) : there is a *construction* of an s (in the domain over which one quantifies) such that P(s) is *proven*
- □x P(x) : there is a *proof* of which it is shown that this specializes to a *proof* of P(s) for every s in the domain of quantification.

J.-J. Ch. Meyer

Intuitionistic logic

- Formal proof system: e.g. 'classical' system of natural deduction *minus the rule of elimination of double negation:*

  ¬¬□
  □

- The rest of the system is the same as that for classical logic, including the *'ex falso sequitur quodlibet'* rule. Remarkably also the rules for the quantifiers □ and □ are the same as in the classical case *despite their different (constructive) interpretation!*

J.-J. Ch. Meyer

Intuitionistic logic

- Some well-known *non* derivable formulas:
  1. ⊬ intuit ¬¬□ □ □
  2. ⊬ intuit □ ¬□
  3. ⊬ intuit □y(□x □ □ □[y/x]) 'Plato's law
- but we *do* have:
- ⊢ intuit □x □ □ there exists term t s.t. ⊢ intuit □[t/x])

J.-J. Ch. Meyer

Axiomatising mathematics

- Followers of *formalism*, like Hilbert, aimed at *a complete axiomatisation of mathematics*
- In particular they thought how *arithmetic* could be axiomatised in a complete way since this constituted the core of mathematics
  - *Peano's axiomatic arithmetic (PA)*

J.-J. Ch. Meyer

Peano's axiomatic arithmetic (PA)

- axioms of PA:
- 1. □x ¬(0 = sx)
- 2. □x, y (sx = sy) □ (x = y)
- 3. □x x + 0 = x
- 4. □x, y x + sy = s(x + y)
- 5. □x, y x □ sy = (x □ y) + x
- 6. □x x □ 0 = 0
- I. (P(0) □ □x (P(x) □ P(sx))) □ □x P(x)

  induction scheme

J.-J. Ch. Meyer

Soundness of PA

- *soundness* of Peano's arithemetic:

  ⊢ PA □ □ N ⊨ □

- where N stands for the *standard model* of de arithmetic, i.e. the *natural numbers with the usual definition of addition, multiplication, successor and equality.*

J.-J. Ch. Meyer

Incompleteness of PA

- Hilbert's programme: there exists a formal system for mathematics that is consistent and complete, in particular there exist such a formal system for arithmetic.
- Kurt Gödel (1931): *'the system PA is incomplete!'*:

  N ⊨ □ ⊭ ⊢ PA □

J.-J. Ch. Meyer

Incompleteness arithmetic

- *Gödel even showed that any sound, consistent formal system incorporating arithmetic, is incomplete: there are always true assertions (that cannot be proven within such a system.*
- final blow for Hilbert's programme!

Presburger arithmetic

- Remarkably, arithmetic on *the natural numbers with only addition and successor (no multiplication) can be completely* axiomatized: *"Presburger arithmetic" is complete.*

Gödel's incompleteness theorems

- Gödel proved the following theorems:

  1. Exist assertion A s.t. $\nvdash_{PA}$ A and $\nvdash_{PA}$ ¬A.

  (1st theorem)

  2. $\nvdash_{PA}$ Con_{PA}, if PA is consistent:

  the consistency of PA is *not* provable *within* PA

  (2nd theorem)

Gödel's theorems

- In fact Gödel proved:
  - there is *no effectively enumerable axiom system* that proves exactly the true (w.r.t. the standard model) arithmetical assertions

Gödel's theorems

- Wij will prove the following proposition:

  3. Exist assertion A s.t. N ⊨ A and $\nvdash_{PA}$ A.
- Note that 1. Follows from 3. :
  - an proposition is not both true and false:
    - not: N ⊨ A and N ⊨ ¬A,
    - I.e. N ⊭ A or N ⊭ ¬A,
    - I.e. N ⊨ A □ N ⊭ ¬A.
  - Furthermore by soundness:
    - $\vdash_{PA}$ A □ N ⊨ A, and so N ⊭ A □ $\nvdash_{PA}$ A.
  - So from 3. □A s.t. N ⊨ A and $\nvdash_{PA}$ A it follows that □A zdd N ⊭ ¬A and $\nvdash_{PA}$ A and so 1.□A zdd $\nvdash_{PA}$ ¬A and $\nvdash_{PA}$ A. ■

Sketch of proof

- Proof. We assume a *computable injective function* (gödel number):

  g: Formulas* □ N

- (i.e. (sequences) formulas in the object language are *uniquely* coded)

Gödel coding

- *arithmetisation of meta-mathematics*
- coding alphabet, e.g.

g(0) = 1	*g*(=) = 5	*g*( ) = 9
g(s) = 2	*g*(x) = 6	*g*(□) = 10
g(+) = 3	*g*(y) = 7	*g*(() = 11
g(□) = 4	*g*(¬) = 8	*g*()) = 12

Gödel coding(2)

- coding formulas, e.g.

  g(x + sy = s(x + y)) =

  $2^6 \cdot 3^3 \cdot 5^2 \cdot 7^7 \cdot 11^5 \cdot 13^2 \cdot 17^{11} \cdot 19^6 \cdot 23^3 \cdot 29^7 \cdot 31^{12}$

- coding sequences of formulas, e.g.

  g($F_1, F_2, F_3, \ldots$) = $2^{g(F_1)} \cdot 3^{g(F_2)} \cdot 5^{g(F_3)} \cdot \ldots$

Meta language □ object language

- *meta-math. assertions* □ *arithmetic assertions*
- E.g. sequence X is prefix of sequence Y □ *g*(X) is a special kind of divisor of *g*(Y)
- a *proof* is a *sequence of formulas* that satisfies a number of *conditions,* which can all be coded in the *object language (i.e. formal arithmetic)*: this is by no means trivial!
- Proof(x, y, z) is expressible as *arithmetic* assertion

Proof predicate

- Define a predicate Proof(x, y, z) s.t.
- Proof(x, y, z) □

  x = *g*(Y) where Y is a proof (sequence of formulas) of a formula F[z] for a formula F with 1 free variable s.t. *g*(F) = y.

- *Proof(x, y, z) can be expressed in the object language by a formula with variables x, y and z.*

"I am not provable!"

- Now consider formula ¬□x Proof(x, y, y). This is a formule with 1 free variable (y) and has gödel number g = *g*(¬□x Proof(x, y, y)).
- Claim: A = ¬□x Proof(x, g, g) satisfies the requirement.
  - note that this formula A = ¬□x Proof(x, g, g) says something like "I am not provable")

Sketch of proof (ctd)

- Claim: N ⊨ ¬□x Proof(x, g, g) and $\nvdash_{PA}$ ¬□x Proof(x, g, g).
- Proof: suppose $\vdash_{PA}$ ¬□x Proof(x, g, g).(*) Then there is a gödel number p of the proof P of ¬□x Proof(x, g, g). So by definition Proof(p, g, g) is true. (p = *g*(P) where P is a proof (sequence of formulas) of a formula F[g] for a formula F with 1 free variable s.t. *g*(F) = g and F = ¬□x Proof(x, y, y)).

Q.E.D.!

- However, from N ⊨ Proof(p, g, g) it follows that N ⊨ □x Proof(x, g, g). Now from the soundness of PA and (*) we obtain that N ⊨ ¬□x Proof(x, g, g), and so N ⊭ □x Proof(x, g, g). Contradiction.

□ (*) is not true: $\nvdash_{PA}$ ¬□x Proof(x, g, g). Consequently N ⊨ ¬□x Proof(x, g, g) !

Q.E.D ■